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INVARIANT BOUNDARY-VALUE PROBLEMS OF AN OPTIMALLY CONTROLLED BOUNDARY LAYER

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The group properties of equations of the variational problem on finding a continuous law for velocity distribution of liquid injection into an incompressible laminar boundary layer in a planar case, which ensures the minimum friction force acting on the airfoil, are considered. It is shown that the optimal injection velocity on a wedge with $x = 0$ is finite.

Key words: *viscous liquid, optimal control, transformation groups.*

In viscous liquid or gas flows, an important problem is the reduction of the drag force, which, for streamlined bodies, is mainly determined by the total friction drag force. One method of solving this problem if the control of local gradients of the streamwise velocity on the wetted surface by means of injection of a liquid into the boundary layer. Since the energy resources (total flow rate of the liquid, power of the control system) are limited, there arises the problem of optimal boundary-layer control, which was first posed in [1]. In [2], the first integral for the conjugate system with respect to the Lagrange multipliers was found using the theory of invariant variational problems and Lie–Ovsyannikov infinitesimal apparatus [3, 4]. In [5], the group properties of equations of planar steady motion of a viscous incompressible liquid in an optimally controlled boundary layer were considered [1, 6]. In the present work, we performed a group classification of these equations, identified classes of invariant boundary-value problems for the case of the power distribution of velocity at the boundary-layer edge, and constructed the corresponding self-similar solutions.

According to [6, 7], finding the law for velocity distribution of liquid injection into a laminar boundary layer, providing the minimum value of the total friction force acting on the airfoil

$$X_{\mathrm{fr}} = \int\limits_0^{x_{\mathrm{end}}} \mu\Big(\frac{\partial u}{\partial y}\Big)_{y=0}\,dx,$$

for a given power of the control system

$$N = \int\limits_0^{x_{\mathrm{end}}} v_{\mathrm{w}}^2(x)\,dx$$

reduces to joint integration of the following equations:

— the Prandtl equations

$$u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} = U_{\mathrm{e}}\,\frac{dU_{\mathrm{e}}}{dx} + \nu\,\frac{\partial^2 u}{\partial y^2}, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

with the boundary conditions

$$u(0,y) = u_0(y) \tag{2}$$

$$u(x,0) = 0, \quad v(x,0) = v_{\mathrm{w}}(x), \quad u(x,\infty) = U_{\mathrm{e}}(x); \tag{3}$$

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— the Euler–Lagrange–Ostrogradskii equations

$$v\,\frac{\partial\lambda_1}{\partial y}+\lambda_1\,\frac{\partial v}{\partial y}+\nu\,\frac{\partial^2\lambda_1}{\partial y^2}+\frac{\partial\lambda_2}{\partial x}+u\,\frac{\partial\lambda_1}{\partial x}=0,\qquad \lambda_1\,\frac{\partial u}{\partial y}-\frac{\partial\lambda_2}{\partial y}=0 \tag{4}$$

with the boundary conditions

$$\lambda_1(x,0)=-\rho,\qquad \lambda_1(x,\infty)=\lambda_2(x,\infty)=0,\qquad \lambda_1(x_{\text{end}},y)=0\quad (y>0),\qquad \lambda_2(x_{\text{end}},y)=0.$$

Here u and v are the projections of the velocity vector onto the x axis directed along the body contour and the y axis directed along the external normal to the wetted surface, respectively, x_{end} is the abscissa of the end of the injection section, ρ, ν, and μ are the density and kinematic and dynamic viscosity of the liquid, the subscripts “e” and “w” refer to quantities at the boundary-layer edge and on the body surface, $u_0(y)$ is the initial velocity profile, $U_{\rm e}(x)$ and $u_0(y)$ are specified functions, and λ_1, λ_2, and α are the Lagrangian multipliers.

The optimal control is found using the formula $v_{\rm w}(x)=\alpha\lambda_2(x,0)$.

We use the first integral of the conjugate system (4), which was found in [2]. Using this integral, system (4) can be replaced by a system of the form

$$u\,\frac{\partial\lambda_2}{\partial x}-\Big(u\,\frac{\partial u}{\partial x}-U_{\rm e}\,\frac{dU_{\rm e}}{dx}\Big)\lambda_1+\nu\,\frac{\partial u}{\partial y}\,\frac{\partial\lambda_1}{\partial y}=0,\qquad \lambda_1\,\frac{\partial u}{\partial y}-\frac{\partial\lambda_2}{\partial y}=0$$

equivalent to one second-order parabolic equation

$$u\,\frac{\partial\lambda_2}{\partial x}-\Big(\nu\,\frac{\partial^2 u}{\partial y^2}+u\,\frac{\partial u}{\partial x}-U_{\rm e}\,\frac{dU_{\rm e}}{dx}\Big)\Big(\frac{\partial u}{\partial y}\Big)^{-1}\frac{\partial\lambda_2}{\partial y}=-\nu\,\frac{\partial^2\lambda_2}{\partial y^2} \tag{5}$$

with the initial condition

$$\lambda_2(x_{\text{end}},y)=0\qquad (y>0) \tag{6}$$

and boundary conditions

$$\frac{\partial\lambda_2(x,0)}{\partial y}=-\rho\,\frac{\partial u(x,0)}{\partial y},\qquad \lambda_2(x,\infty)=0. \tag{7}$$

We find the main group of continuous transformations [3, 4] admitted by system (1), (5). In the case considered, the problem of group analysis reduces to the problem of group classification with respect to the arbitrary element $\theta(x)\equiv U_{\rm e}\,dU_{\rm e}/dx$. The groups of transformations admitted by system (1), (5) for various specializations of the arbitrary element are determined by their Lie algebras of infinitesimal operators of the form

$$X=\xi_x\,\frac{\partial}{\partial x}+\xi_y\,\frac{\partial}{\partial y}+\xi_u\,\frac{\partial}{\partial u}+\xi_v\,\frac{\partial}{\partial v}+\xi_{\lambda_2}\,\frac{\partial}{\partial\lambda_2}.$$

Constructing the constitutive equations and their common solution, we obtain

$$\xi_x=A_1x+A_2,\quad \xi_y=B_1y+B(x),\quad \xi_u=(A_1-2B_1)u,\quad \xi_v=-B_1v+u\,\frac{dB}{dx},\quad \xi_{\lambda_2}=D_1\lambda_2+D_2, \tag{8}$$

where A_1, A_2, B_1, D_1, and D_2 are constants, and $B(x)$ is an arbitrary function. The constants A_1, A_2, and B_1 are related to the arbitrary element $\theta(x)$ by the constitutive equation

$$(A_1x+A_2)\frac{d\theta}{dx}=(A_1-4B_1)\theta, \tag{9}$$

which plays the role of a classifying equation.

For the arbitrary function $\theta(x)$, Eq. (9) is satisfied only if $A_1=A_2=B_1=0$.

Hence, the kernel of the main Lie algebras of system (1), (5) is formed by the operators

$$X_1=\lambda_2\,\frac{\partial}{\partial\lambda_2},\quad X_2=\frac{\partial}{\partial\lambda_2},\quad X_\infty=B\,\frac{\partial}{\partial y}+u\,\frac{dB}{dx}\,\frac{\partial}{\partial v}.$$

The results of group classification are listed in Table 1 ($X_3=\partial/\partial x$, $X_4=x\,\partial/\partial x+u\,\partial/\partial u$, and $X_5=y\,\partial/\partial y-2u\,\partial/\partial u-v\,\partial/\partial v$).

Further, we consider the power distribution of velocity at the boundary-layer edge $U_{\rm e}=c_0x^m$.

Let $m\neq 0$. In this case, for the boundary conditions (3), (7) to be invariant with allowance for (9), the following conditions should be satisfied:

$$B=A_2=0,\qquad 2B_1=(1-m)A_1,\qquad D_1=A_1-2B_1,\qquad D_2=0,$$
$$A_1(v'_{\rm w}(x)+(1-m)v_{\rm w}(x)/(2x))=0. \tag{10}$$

TABLE 1

$\theta(x)$	Operators
0	$X_3,\ X_4,\ X_5$
$\pm e^x$	$4X_3 - X_5$
1	$X_3,\ 4X_4 + X_5$
$\pm x^n$	$4X_4 - (n-1)X_5,\ n \neq 0$

The latter relation for the arbitrary function $v_w(x)$ is satisfied for $A_1 = 0$, which corresponds to the trivial case of an identical transformation. Assuming that $A_1 \neq 0$, we obtain $v_w(x) = cx^{(m-1)/2}$.

Thus, for the boundary-value problem (1), (3), (5), (7) to be an invariant boundary-value problem [8], relations (8) and (10) should be satisfied. If the latter is true, the boundary-value problem is written in terms of invariants of the group represented by the operator

$$X = x\,\frac{\partial}{\partial x} + \frac{1-m}{2}\,y\,\frac{\partial}{\partial y} + mu\,\frac{\partial}{\partial u} + \frac{m-1}{2}\,v\,\frac{\partial}{\partial v} + m\lambda_2\,\frac{\partial}{\partial \lambda_2}. \qquad (11)$$

Let $m = 0$. For invariance of the boundary conditions (3) and (7), the following relations should be satisfied:

$$B = 0, \quad 2B_1 = A_1, \quad D_1 = D_2 = 0, \quad (A_1x + A_2)v_w'(x) + A_1v_w(x)/2 = 0.$$

For $A_1 \neq 0$, the latter equation yields $v_w(x) = c/\sqrt{A_1x + A_2}$. The group is represented by the operators

$$X = x\,\frac{\partial}{\partial x} + \frac{y}{2}\,\frac{\partial}{\partial y} - \frac{v}{2}\,\frac{\partial}{\partial v}, \qquad X = \frac{\partial}{\partial x},$$

the first of them being a particular case of operator (11) for $m = 0$.

In both cases considered, the invariants of the initial condition (6) is observed for $A_1 = 0$ only; hence, it is impossible to find any invariant solutions of rank 1 [3] satisfying all the boundary conditions. If the initial condition (6) is set for $x = 0$, it is invariant with respect to transformations corresponding to operator (11), and the variational problem (1)–(3), (5)–(7) admits a self-similar solution by virtue of automatic satisfaction of the initial condition (2) for the Prandtl equations [9]. Thus, in constructing invariant solutions, we do not require invariance of condition (6).

The invariant solution $u = u(y)$, $v = v(y)$, $\lambda_2 = \lambda_2(y)$ corresponding to the transfer operator $X = \partial/\partial x$ allows obtaining an exact solution of the Prandtl equations (1) for uniform suction of the liquid from the boundary layer

$$v(y) = v_0 = \text{const} < 0, \qquad u(y) = U_e[1 - \exp(v_0y/\nu)],$$

which is an exact solution of the Navier–Stokes equations in the planar case [10], and also a solution of the conjugate equation (5) $\lambda_2(x) = \rho U_e \exp(v_0y/\nu)$, which satisfies the boundary conditions (7).

The invariant solution constructed using operator (11) admitted for all values of the parameter m can be written as

$$u = c_0x^m\Phi'(\eta), \qquad v = -\sqrt{\frac{2\nu c_0}{m+1}}\,x^{(m-1)/2}\Big(\frac{m-1}{2}\,\eta\Phi' + \frac{m+1}{2}\,\Phi\Big),$$
$$\lambda_2 = c_1x^m g(\eta), \qquad \eta = \sqrt{\frac{m+1}{2}\,\frac{c_0}{\nu}}\,yx^{(m-1)/2}. \qquad (12)$$

The boundary-value problem (1), (3), (5), (7) on the solution of the form (12) can be reduced to the following boundary-value problem for a system of ordinary differential equations:

$$\Phi''' + \Phi\Phi'' + \beta(1 - \Phi'^2) = 0, \qquad \beta = 2m/(m+1); \qquad (13)$$

$$\Phi(0) = c_w, \quad \Phi'(0) = 0, \quad \Phi'(\infty) = 1; \qquad (14)$$

$$\Phi'''g' - \Phi''g'' - \beta\Phi'\Phi''g - \beta(1 - \Phi'^2)g' = 0; \qquad (15)$$

$$g'(0) = -\rho c_0\Phi''(0)/c_1, \qquad g(\infty) = 0. \qquad (16)$$

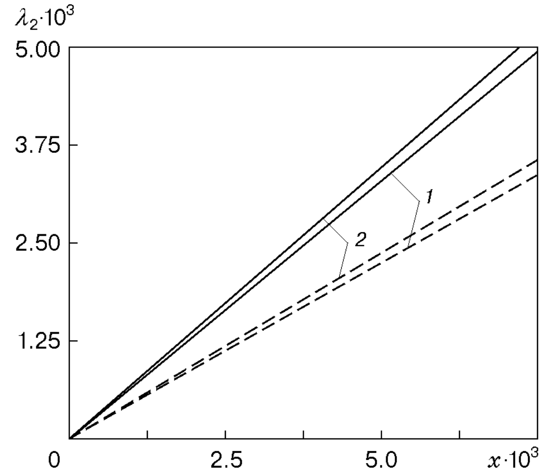

Fig. 1. Lagrangian multiplier λ_2 as a function of the streamwise coordinate x for $m = 1$ and $y = 0$ (solid curves) and $y = 0.2$ (dashed curves); curves 1 and 2 show the self-similar solution and the difference solution, respectively.

The solution of the Falkner–Skan equation (13) under the boundary conditions (14) with $c_{\rm w} = 0$ is known as the Hartree solution [11]. Using this solution, we find the solution of the boundary-value problem (15), (16).

In (12), (15), and (16), we pass to dimensionless quantities: $\bar{x} = x/l$, $\bar{y} = y\sqrt{\mathrm{Re}_r}/l$, $\mathrm{Re}_r = U_r l/\nu$, $U_r = U_{\rm e}(l)$, $\bar{u} = u/U_r$, $\bar{v} = v\sqrt{\mathrm{Re}_r}\,/U_r$, $\bar{\lambda}_2 = \lambda_2/(\rho U_r)$, and $\bar{g} = c_1 g/(\rho c_0)$. Then, relations (15) and (16) are written in the following form (hereinafter, the bar over dimensionless quantities is omitted):

$$\Phi''' g' - \Phi'' g'' - \beta\Phi'\Phi'' g - \beta(1 - \Phi'^2) g' = 0; \tag{17}$$

$$g'(0) = -\Phi''(0), \qquad g(\infty) = 0. \tag{18}$$

As a result of solving the two-point boundary-value problem (17), (18) by the shooting method, we obtained the values $g(0) = 1$, 0.75649, 0.70304, 0.67965, 0.66656, and 0.65821 for $m = 0$, 0.2, 0.4, 0.6, 0.8, and 1.0, respectively.

Figure 1 shows the dependences

$$\lambda_2(x, y) = x^m g(\eta) \tag{19}$$

for the case $m = 1$, $c_{\rm w} = 0$ with fixed values of y and the dependences $\lambda_2(x, y)$ obtained by solving the initial-boundary problem (1)–(3), (5)–(7) with $v_{\rm w}(x) = 0$ by the difference method for the same values of m and y. It follows from Fig. 1 that the self-similar solution (19), which obviously does not satisfy the initial condition (6), in the vicinity of the point $x = 0$ is close to the exact solution obtained by the difference method and infinitely approaches the latter as $x \to 0$. Note, for $m = 1$, the self-similar solution (19) has no singularity in the origin.

Figure 2 shows the self-similar solution (19) for $c_{\rm w} = 0$ and the exact difference solution for $v_{\rm w}(x) = 0$. A comparison of these solutions shows that the self-similar solution approximates the exact solution for values of x close to zero and almost coincides with the exact solution already at $x = 0.004$. This conclusion does not refer to the value $x = 0$, since the self-similar solution in the case considered has a singularity in the origin. Nevertheless, for $m = 0$, Eq. (17) is solvable in a finite form with respect to $g(\eta)$: $g(\eta) = a\Phi'(\eta) + b$, where a and b are integration constants. We find a and b by satisfying the boundary conditions (18): $a\Phi''(0) = -\Phi''(0)$ and $b = -a$. For $\Phi''(0) \neq 0$, we obtain $a = -1$ and $b = 1$ [for $\Phi''(0) = 0$, the friction stress on the wall vanishes, and the problem of friction minimization makes no sense]. Thus, we have $g(\eta) = 1 - \Phi'(\eta)$ and, hence,

$$\lambda_2 = 1 - \Phi'(\eta)$$

or, in a dimensional form,

$$\lambda_2 = \rho(U_\infty - u). \tag{20}$$

Note, this expression with $m = 0$ in the general case of an arbitrary structure of the function $u(x, y)$ is an exact analytical solution of Eq. (5), satisfying the boundary conditions (7).

It is impossible to obtain a numerical solution of the initial-boundary problem (5)–(7) for the conjugate equation at the point $x = 0$ for $m = 0$, which is caused by a singularity present at this point in the solution of

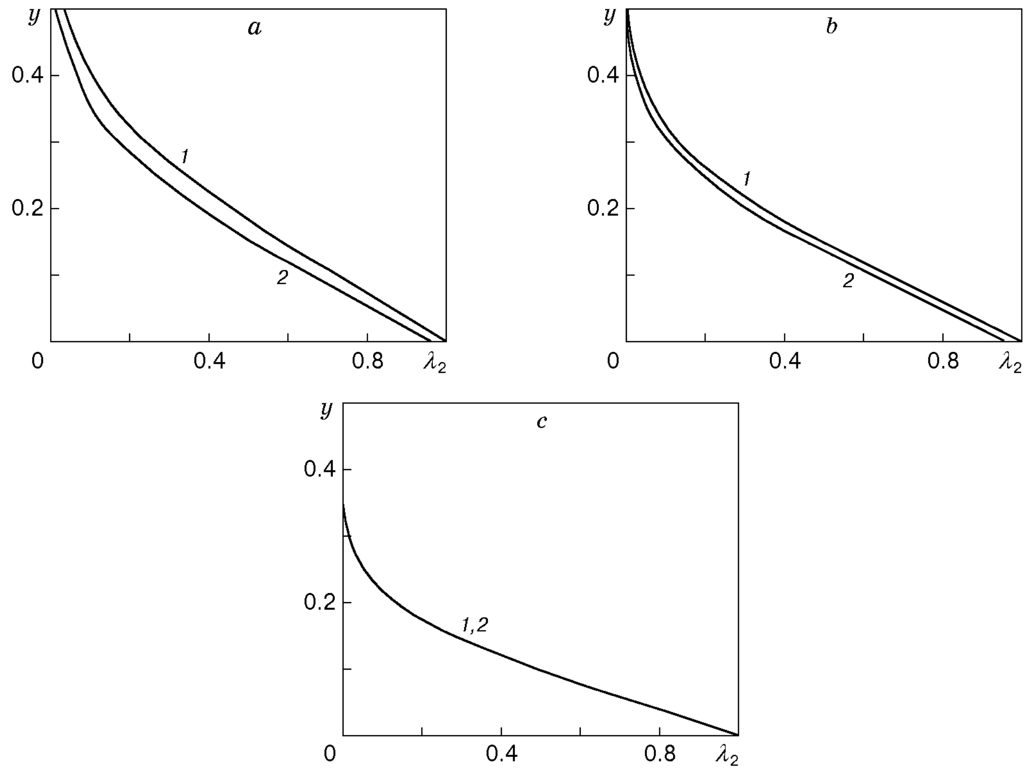

Fig. 2. Lagrangian multiplier λ_2 as a function of the transverse coordinate y for $m = 0$ and $x = 0.014$ (a), 0.009 (b), and 0.004 (c); curves 1 and 2 refer to the self-similar solution and difference solution, respectively.

the Prandtl equations. A comparison of this solution with the self-similar solution in the vicinity of the point $x = 0$ (for $m = 1$, also at the point $x = 0$ itself) and an analysis of solution (20) allow us to conclude that the function λ_2 and, hence, $v_{\rm w}(x)$ take finite values at $x = 0$, namely, $\lambda_2(0, 0) = 1$ and $v_{\rm w}(0) = \rho\sqrt{{\rm Re}_r}\,\alpha$ for $m = 0$ and $\lambda_2(0, 0) = v_{\rm w}(0) = 0$ for $m \in (0; 1]$. This result is principally important for implementation of the optimal injection law and correct satisfaction of the restriction on the control-system power.

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